Convex Optimization & Machine Learning

Convex Functions

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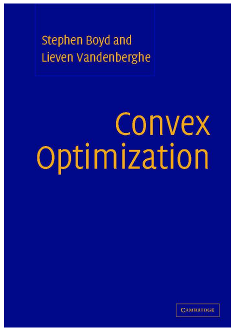
Most slides in this lecture are taken from 

Quizz

if C and C' are two **convex** sets, then $C \cup C'$ is **convex**?

1. **Yes** 2. **No**

Quizz

if C and C' are two convex sets, then $C \cup C'$ is convex?

1. **Yes** 2. **No**

if C_i is a family of **convex** sets, then $\bigcap_{i \in I} C_i$ is **convex**

Quizz

The matrix $M = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 2 & 8 \end{bmatrix}$ is **positive definite?**

1. **Yes** 2. **No**

Quizz

The matrix $M = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 5 & \mathbf{3} \\ 1 & \mathbf{2} & 8 \end{bmatrix}$ is **positive definite?**

1. **Yes** 2. **No**

has to be symmetric at least...

Quizz

The matrix $M = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 8 \end{bmatrix}$ is **positive definite?**

1. **Yes** 2. **No**

Solving optimization problems

general optimization problem

- very difficult to solve
- methods involve some compromise, $e.g.$, very long computation time, or not always finding the solution

exceptions: certain problem classes can be solved efficiently and reliably

- least-squares problems
- linear programming problems
- **convex** optimization problems

Convex functions

Short outline

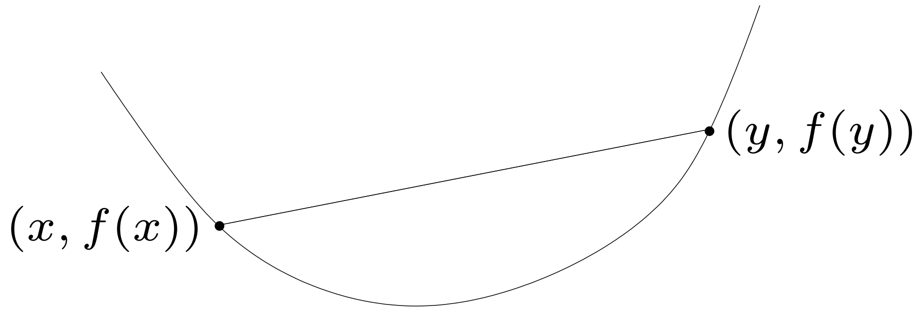
- basic properties and examples
- operations that preserve convexity
- the conjugate function

Definition

$f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathbf{dom}\, f$ is a convex set and

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

for all $x, y \in \mathbf{dom}\, f$, $0 \leq \theta \leq 1$



- f is concave if $-f$ is convex
- f is strictly convex if $\mathbf{dom}\, f$ is convex and

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta) f(y)$$

for $x, y \in \mathbf{dom}\, f$, $x \neq y$, $0 < \theta < 1$

Examples on $\mathbb{R}$

convex:

- affine: $ax + b$ on $\mathbb{R}$, for any $a, b \in \mathbb{R}$
- exponential: e^{ax}, for any $a \in \mathbb{R}$
- powers: x^α on $\mathbb{R}_{++}$, for $\alpha \geq 1$ or $\alpha \leq 0$
- powers of absolute value: $|x|^p$ on $\mathbb{R}$, for $p \geq 1$
- negative entropy: $x \log x$ on $\mathbb{R}_{++}$

concave:

- affine: $ax + b$ on $\mathbb{R}$, for any $a, b \in \mathbb{R}$
- powers: x^α on $\mathbb{R}_{++}$, for $0 \leq \alpha \leq 1$
- logarithm: $\log x$ on $\mathbb{R}_{++}$

Quizz

A **norm** is a function $\|\cdot\|$ that satisfies

- $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t| \, \|x\|$ for $t \in \mathbb{R}$
- $\|x + y\| \leq \|x\| + \|y\|$

A norm is a convex function?

1. **Yes** 2. **No**

Quizz

A **norm** is a function $\|\cdot\|$ that satisfies

- $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t| \, \|x\|$ for $t \in \mathbb{R}$
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A norm is a convex function?

1. **Yes** 2. **No**

Examples on $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$

affine functions are convex and concave; all norms are convex

examples on $\mathbb{R}^n$

- affine function $f(x) = a^T x + b$
- norms: $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$; $\|x\|_\infty = \max_k |x_k|$

examples on $\mathbb{R}^{m\times n}$ ($m \times n$ matrices)

- affine function

$$f(X) = \mathbf{Tr}(A^T X) + b = \sum_{i=1}^m \sum_{j=1}^n A_{ij} X_{ij} + b$$

- spectral (maximum singular value) norm

$$f(X) = \|X\|_2 = \sigma_{\max}(X) = (\lambda_{\max}(X^T X))^{1/2}$$

Restriction of a convex function to a line

$f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if the function $g : \mathbb{R} \to \mathbb{R}$,

$$g(t) = f(x + tv), \qquad \mathbf{dom}\, g = \{t \mid x + tv \in \mathbf{dom}\, f\}$$

is convex (in t) for any $x \in \mathbf{dom}\, f$, $v \in \mathbb{R}^n$

can check convexity of f by checking convexity of functions of **one** variable

Restriction of a convex function to a line

example. $f : \mathbf{S}^n \to \mathbb{R}$ with $f(X) = \log\det X$, $\mathbf{dom}\, X = \mathbf{S}^n_{++}$

$$\begin{aligned} g(t) = \log\det(X + tV) &= \log\det X + \log\det(I + tX^{-1/2}VX^{-1/2}) \\ &= \log\det X + \sum_{i=1}^{n} \log(1 + t\lambda_i) \end{aligned}$$

where λ_i are the eigenvalues of $X^{-1/2}VX^{-1/2}$

g is concave in t (for any choice of $X \succ 0$, V); hence f is concave

First-order condition

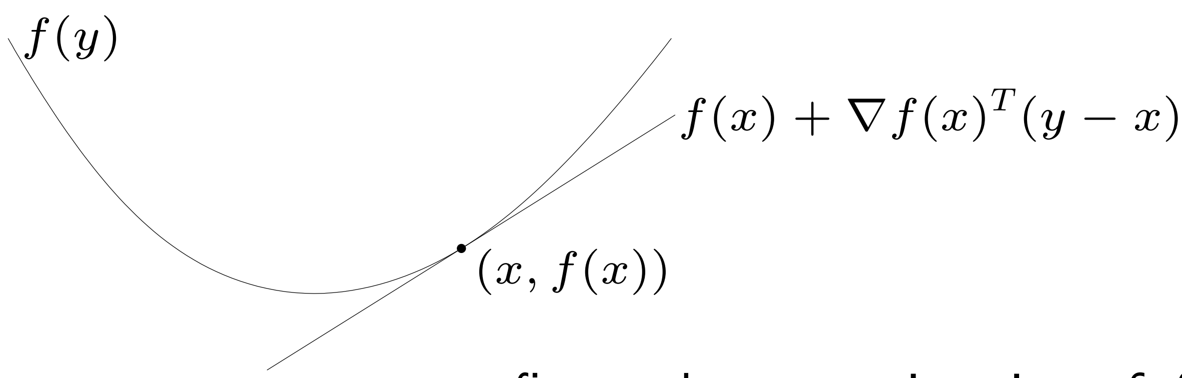
f is **differentiable** if $\mathbf{dom}\, f$ is open and the gradient

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \dots, \frac{\partial f(x)}{\partial x_n} \right)$$

exists at each $x \in \mathbf{dom}\, f$

1st-order condition: differentiable f with convex domain is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \text{for all } x, y \in \mathbf{dom}\, f$$



first-order approximation of f is global **underestimator**.

Second-order conditions

f is **twice differentiable** if $\mathbf{dom}\, f$ is open and the **Hessian** $\nabla^2 f(x) \in \mathbf{S}^n$,

$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n,$$

exists at each $x \in \mathbf{dom}\, f$

2nd-order conditions: for twice differentiable f with convex domain

- f is **convex** if and only if

$$\nabla^2 f(x) \in \mathbf{S}^n_+ \quad \text{for all } x \in \mathbf{dom}\, f$$

- if $\nabla^2 f(x) \succ 0$ (also written $\nabla^2 f(x) \in \mathbf{S}^n_{++}$) for all $x \in \mathbf{dom}\, f$, then f is **strictly convex**

Examples

quadratic function: $f(x) = (1/2)x^T Px + q^T x + r$ (with $P \in \mathbf{S}^n$)

$$\nabla f(x) = Px + q, \qquad \nabla^2 f(x) = P$$

convex if $P \succeq 0$

least-squares objective: $f(x) = \|Ax - b\|_2^2$

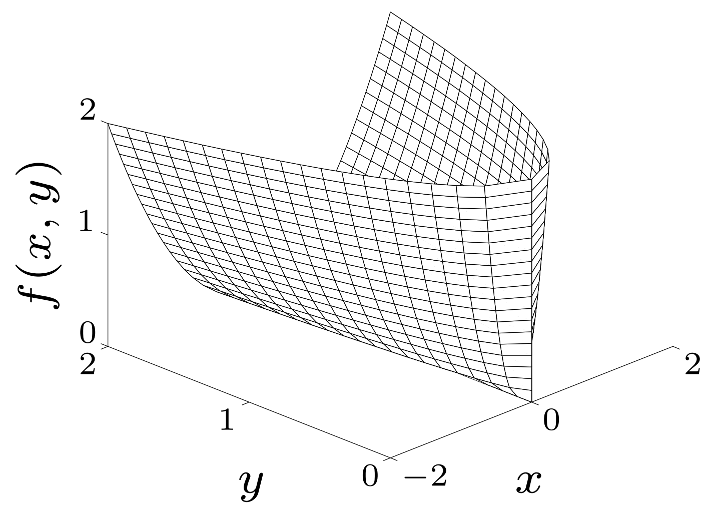
$$\nabla f(x) = 2A^T(Ax - b), \qquad \nabla^2 f(x) = 2A^T A$$

convex (for any A)

quadratic-over-linear: $f(x, y) = x^2/y$

$$\nabla^2 f(x, y) = \frac{2}{y^3} \begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix}^T \succeq 0$$

convex for $y > 0$



log-sum-exp: $f(x) = \log \sum_{k=1}^{n} \exp x_k$ is convex

$$\nabla^2 f(x) = \frac{1}{\mathbb{1}_{d,d}^T z} \mathbf{diag}(z) - \frac{1}{(\mathbb{1}_{d,d}^T z)^2} z z^T \qquad (z_k = \exp x_k)$$

to show $\nabla^2 f(x) \succeq 0$, we must verify that $v^T \nabla^2 f(x) v \geq 0$ for all v:

$$v^T \nabla^2 f(x) v = \frac{(\sum_k z_k v_k^2)(\sum_k z_k) - (\sum_k v_k z_k)^2}{(\sum_k z_k)^2} \geq 0$$

since $(\sum_k v_k z_k)^2 \leq (\sum_k z_k v_k^2)(\sum_k z_k)$ (from Cauchy-Schwarz inequality)

geometric mean: $f(x) = (\prod_{k=1}^{n} x_k)^{1/n}$ on $\mathbb{R}_{++}^n$ is concave

(similar proof as for log-sum-exp)

Epigraph and sublevel set

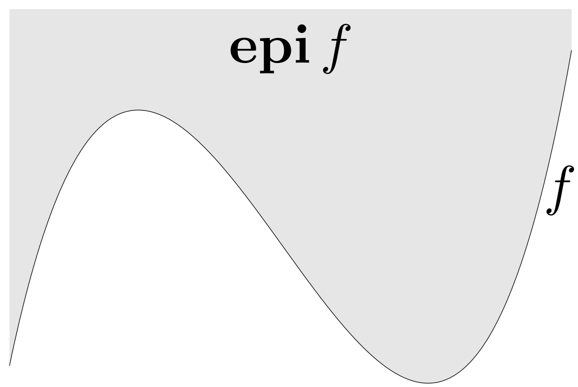
α-**sublevel set** of $f : \mathbb{R}^n \to \mathbb{R}$:

$$C_\alpha = \{x \in \mathbf{dom}\, f \mid f(x) \leq \alpha\}$$

sublevel sets of convex functions are convex (converse is false)

epigraph of $f : \mathbb{R}^n \to \mathbb{R}$:

$$\mathbf{epi}\, f = \{(x, t) \in \mathbb{R}^{n+1} \mid x \in \mathbf{dom}\, f,\ f(x) \leq t\}$$



f is convex if and only if $\mathbf{epi}\, f$ is a convex set

Jensen's inequality

basic inequality: if f is convex, then for $0 \leq \theta \leq 1$,

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$

extension: if f is convex, then

$$f(\mathbf{E}z) \leq \mathbf{E}f(z)$$

for any random variable z

basic inequality is special case with discrete distribution

$$\mathbf{Prob}(z = x) = \theta, \qquad \mathbf{Prob}(z = y) = 1 - \theta$$

Operations that preserve convexity

To recapitulate: what are the practical methods to prove that a function is convex?

1. **verify definition** (often simplified by restricting to a line)

2. for twice differentiable functions, show

$$\nabla^2 f(x) \succeq 0$$

Operations that preserve convexity

To recapitulate: what are the practical methods to prove that a function is convex?

1. **verify definition** (often simplified by restricting to a line)

2. for twice differentiable functions, show

$$\boldsymbol{\nabla^2 f(x) \succeq 0}$$

$\boxed{\text{There's also another } \textbf{a third one}}$

3. show that f is obtained from simple convex functions by **operations that preserve convexity**

Positive weighted sum & composition with affine function

nonnegative multiple: αf is convex if f is convex, $\alpha \geq 0$

sum: $f_1 + f_2$ convex if f_1, f_2 convex (extends to infinite sums, integrals)

composition with affine function: $f(Ax + b)$ is convex if f is convex

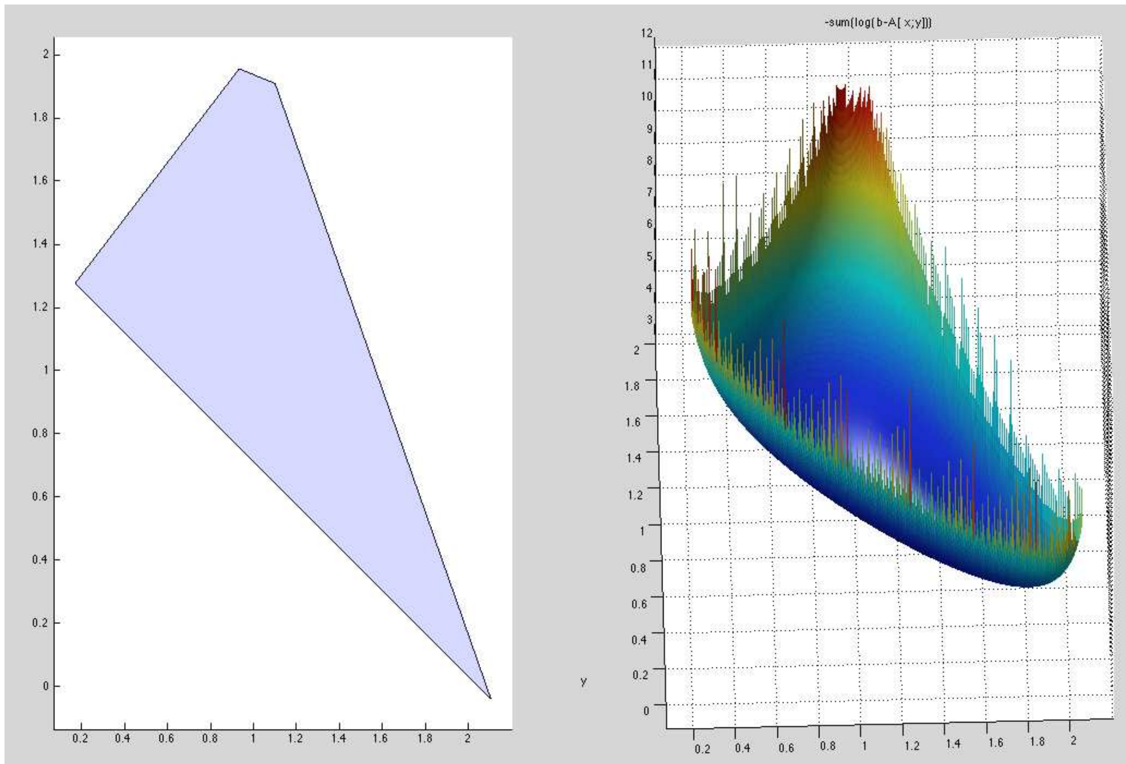
examples

- log barrier for linear inequalities

$$f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x), \qquad \mathbf{dom}\, f = \{x \mid a_i^T x < b_i, i = 1, \ldots, m\}$$

- norm of affine function: $f(x) = \|Ax + b\|$

Log-Barrier



Quizz

if $f_1, \ldots, f_m$ are **convex**, then $f(x) = \min\{f_1(x), \ldots, f_m(x)\}$ is **convex**?

1. **Yes** 2. **No**

if $f_1, \ldots, f_m$ are **concave**, then $f(x) = \min\{f_1(x), \ldots, f_m(x)\}$ is **concave**?

1. **Yes** 2. **No**

Pointwise maximum

if f_1, . . . , f_m are convex, then $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ is convex

examples

- piecewise-linear function: $f(x) = \max_{i=1,\ldots,m}(a_i^T x + b_i)$ is convex
- sum of r largest components of $x \in \mathbb{R}^n$:

$$f(x) = x_{[1]} + x_{[2]} + \cdots + x_{[r]}$$

  is convex ($x_{[i]}$ is ith largest component of x)

  proof:

$$f(x) = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_r} \mid 1 \le i_1 < i_2 < \cdots < i_r \le n\}$$

Pointwise supremum

if $f(x,y)$ is convex in x for each $y \in \mathcal{A}$, then

$$g(x) = \sup_{y \in \mathcal{A}} f(x,y)$$

is convex

examples

- support function of a set C: $S_C(x) = \sup_{y \in C} y^T x$ is convex
- distance to farthest point in a set C:

$$f(x) = \sup_{y \in C} \|x - y\|$$

- maximum eigenvalue of symmetric matrix: for $X \in \mathbf{S}^n$,

$$\lambda_{\max}(X) = \sup_{\|y\|_2 = 1} y^T X y$$

Composition with scalar functions

composition of $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$:

$$f(x) = h(g(x))$$

f is convex if $\begin{array}{l} g \text{ convex, } h \text{ convex, } h \text{ nondecreasing} \\ g \text{ concave, } h \text{ convex, } h \text{ nonincreasing} \end{array}$

- proof (for $n = 1$, differentiable g, h). Can be extended to non-differentiable g, h

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

- note: monotonicity must hold for extended-value extension $\tilde{h}$

examples

- $\exp g(x)$ is convex if g is convex
- $1/g(x)$ is convex if g is concave and positive

Vector composition

composition of $g : \mathbb{R}^n \to \mathbb{R}^k$ and $h : \mathbb{R}^k \to \mathbb{R}$:

$$f(x) = h(g(x)) = h(g_1(x), g_2(x), \ldots, g_k(x))$$

f is convex if
- g_i convex, h convex, h nondecreasing in each argument
- g_i concave, h convex, h nonincreasing in each argument

proof (for $n = 1$, differentiable g, h). Can be extended to non-differentiable g, h

$$f''(x) = g'(x)^T \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))^T g''(x)$$

examples

- $\sum_{i=1}^m \log g_i(x)$ is concave if g_i are concave and positive
- $\log \sum_{i=1}^m \exp g_i(x)$ is convex if g_i are convex

Minimization

if $f(x,y)$ is convex in (x,y) and C is a convex set, then

$$g(x) = \inf_{y \in C} f(x,y)$$

is convex

examples

- $f(x,y) = x^T A x + 2x^T B y + y^T C y$ with

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0, \qquad C \succ 0$$

  minimizing over y gives $g(x) = \inf_y f(x,y) = x^T (A - BC^{-1}B^T) x$

  g is convex, hence Schur complement $A - BC^{-1}B^T \succeq 0$

- distance to a set: $\mathbf{dist}(x,S) = \inf_{y \in S} \|x - y\|$ is convex if S is convex

Perspective

the **perspective** of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the function $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$,

$$g(x,t) = tf(x/t), \qquad \mathbf{dom}\, g = \{(x,t) \mid x/t \in \mathbf{dom}\, f,\ t > 0\}$$

g is convex if f is convex

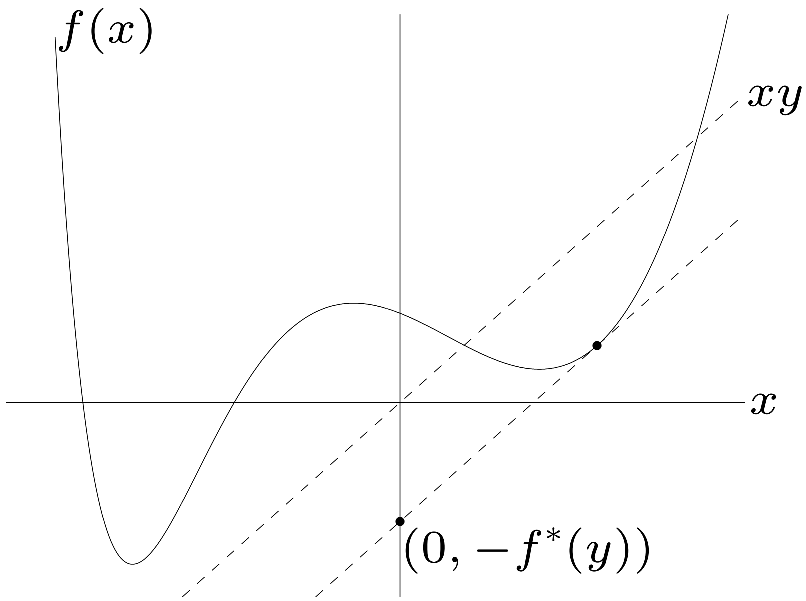
examples

- $f(x) = x^T x$ is convex; hence $g(x,t) = x^T x/t$ is convex for $t > 0$
- negative logarithm $f(x) = -\log x$ is convex; hence relative entropy $g(x,t) = t\log t - t\log x$ is convex on $\mathbb{R}^2_{++}$

The conjugate function

the (Fenchel) **conjugate** of a function f is

$$f^*(y) = \sup_{x \in \mathbf{dom}\, f} (y^T x - f(x))$$



- f^* at y is the maximum gap between yx and f over all possible x.
- f^* is convex (even if f is not)
- will be useful when we study duality.

The conjugate function

examples

- negative logarithm $f(x) = -\log x$

$$
\begin{aligned}
f^*(y) &= \sup_{x>0}(xy + \log x) \\
&= \begin{cases} -1 - \log(-y) & y < 0 \\ \infty & \text{otherwise} \end{cases}
\end{aligned}
$$

- strictly convex quadratic $f(x) = (1/2)x^T Q x$ with $Q \in \mathbf{S}^n_{++}$

$$
\begin{aligned}
f^*(y) &= \sup_{x}(y^T x - (1/2)x^T Q x) \\
&= \frac{1}{2} y^T Q^{-1} y
\end{aligned}
$$

Optimization problem in standard form

Optimization problem in standard form

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p
\end{array}
$$

- $x \in \mathbf{R}^n$ is the optimization variable
- $f_0 : \mathbf{R}^n \to \mathbf{R}$ is the objective or cost function
- $f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, m$, are the inequality constraint functions
- $h_i : \mathbf{R}^n \to \mathbf{R}$ are the equality constraint functions

optimal value:

$$
p^\star = \inf\{f_0(x) \mid f_i(x) \leq 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p\}
$$

- $p^\star = \infty$ if problem is infeasible (no x satisfies the constraints)
- $p^\star = -\infty$ if problem is unbounded below

Optimal and locally optimal points

- x is **feasible** if $x \in \textbf{dom}\, f_0$ and it satisfies the constraints

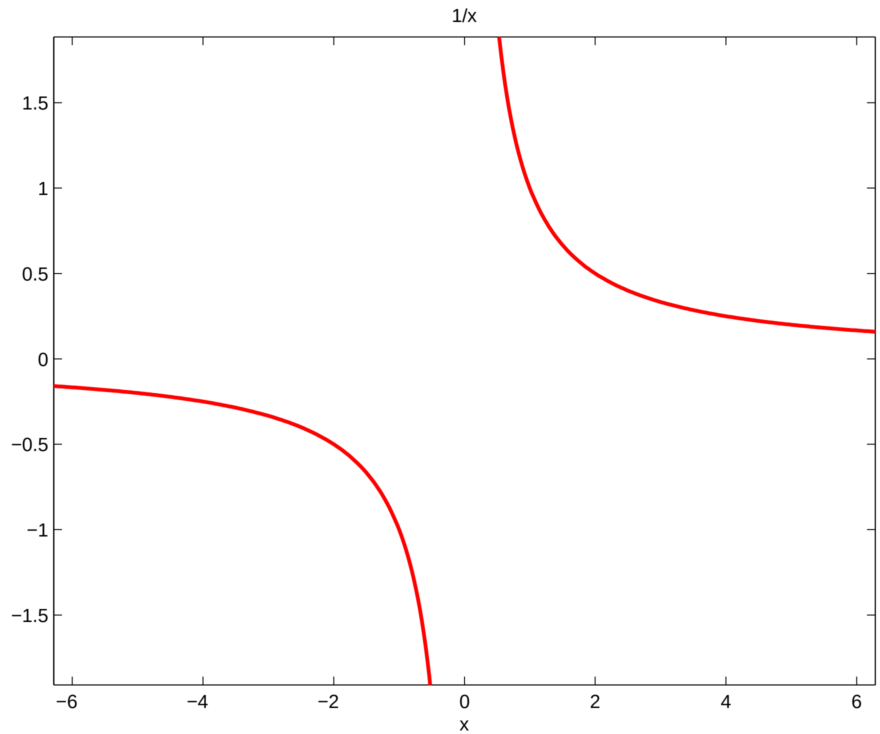
- a feasible x is **optimal** if $f_0(x) = p^\star$;

- X_{opt} is the set of optimal points

- x is **locally optimal** if there is an $R > 0$ such that x is optimal for

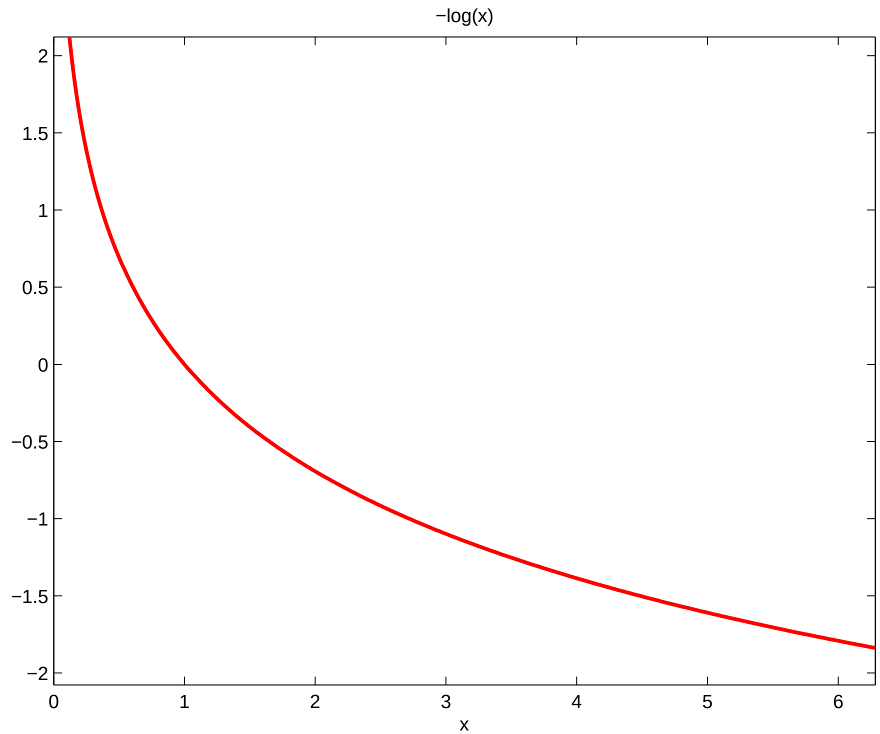
$$\begin{array}{ll} \text{minimize (over } z\text{)} & f_0(z) \\ \text{subject to} & f_i(z) \le 0, \quad i = 1, \dots, m, \quad h_i(z) = 0, \quad i = 1, \dots, p \\ & \|z - x\|_2 \le R \end{array}$$

Examples (with $n = 1$, $m = p = 0$)



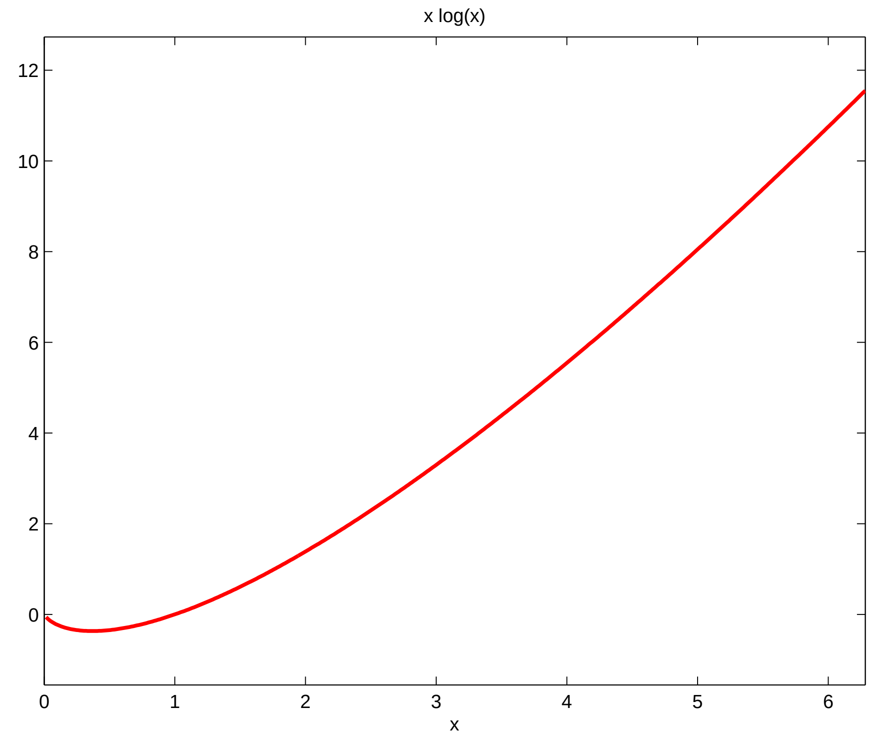
$f_0(x) = 1/x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = 0$, no optimal point

Examples (with $n = 1$, $m = p = 0$)



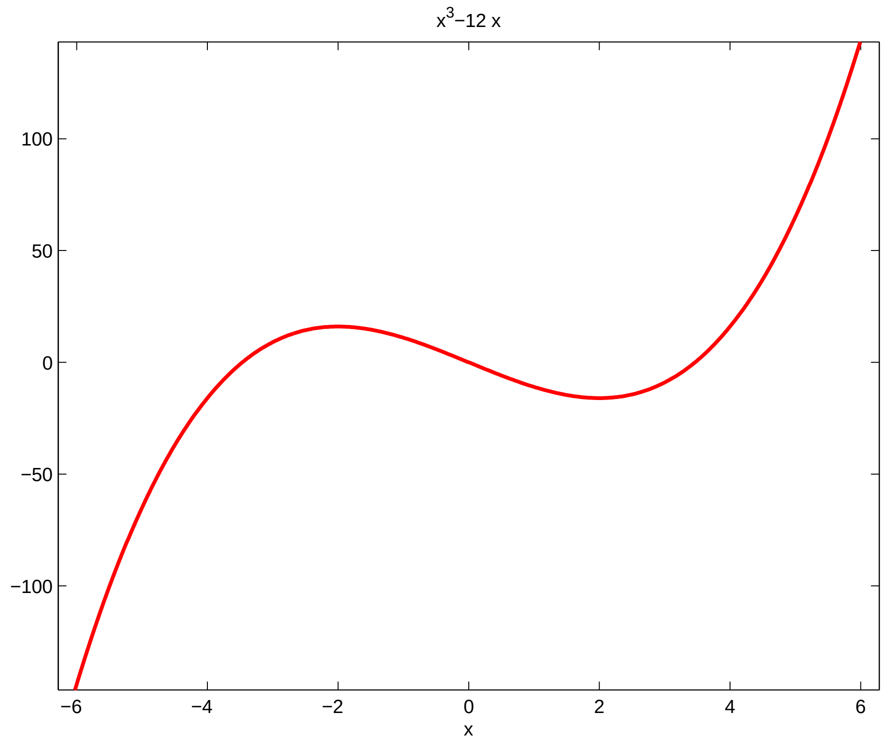
$$f_0(x) = -\log x, \ \mathbf{dom}\, f_0 = \mathbf{R}_{++}\colon \ p^\star = -\infty$$

Examples (with $n = 1$, $m = p = 0$)



$f_0(x) = x \log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = -1/e$, $x = 1/e$ is optimal

Examples (with $n = 1$, $m = p = 0$)



$f_0(x) = x^3 - 12x$, $p^\star = -\infty$, local optimum at $x = 2$

Implicit constraints

the standard form optimization problem has an **implicit constraint**

$$x \in \mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\, f_i \;\cap\; \bigcap_{i=1}^{p} \mathbf{dom}\, h_i,$$

- we call $\mathcal{D}$ the **domain** of the problem
- the constraints $f_i(x) \leq 0$, $h_i(x) = 0$ are the explicit constraints
- a problem is **unconstrained** if it has no explicit constraints $(m = p = 0)$

example:

$$\text{minimize} \quad f_0(x) = -\sum_{i=1}^{k} \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints $a_i^T x < b_i$

Feasibility problem

$$\begin{array}{ll} \textbf{find} & x \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

can be considered a special case of the general problem with $f_0(x) = 0$:

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

- $p^\star = 0$ if constraints are feasible; any feasible x is optimal
- $p^\star = \infty$ if constraints are infeasible